L-REGULAR LINEAR CONNECTIONS

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Introduction

An adequate and interesting approach to the theory of nonlinear connections has been accomplished by Grifone [3]. His definition of a nonlinear connection is based on the geometry of the tangent bundle $T(M)$ of a differentiable manifold M. In his theory, the natural almost-tangent structure J on $T(M)$ ([5] and [8]) plays an extremely important role.

Anona [1] generalized the notion of the natural almost-tangent structure by considering a vector 1-form L on the manifold M—not on $T(M)$—satisfying certain conditions. As a by-product of his work, a generalization of some of Grifone's results was obtained.

The first author of the prsent paper, adopting the point of view of Anona, generalized Grifone's theory of nonlinear connections [10]. Grifone's theory can be retrieved from [10] by letting M be the tangent bundle of a differentiable manifold and L the natural almost-tangent structure J on M.

In this paper, we still adopt the point of view of Anona and continue developing the approach established in [10]. After the notations and preliminaries (§1), the first part (§2) of the work is devoted to the problem of associating to each L-regular linear connection on M a nonlinear L-connection on M. The route we have followed is significantly different from that of Grifone [4]. Following Tamnou [8], we introduce an almost-complex and an almost-product structures on M by means of a given L-regular linear connection on M. The product of these two structures defines a nonlinear L-connection on M, which generalizes Grifone's nonlinear connection [4].

The seconed part (§3) is devoted to the converse problem: associating to each nonlinear L-connection Γ on M an L-regular linear connection on M; called the L-lift of Γ . The existence of this lift is established and the fundamental tensors associated with it are studied.

In the third part (§4), we investigate the L-lift of a homogeneous L-connection Γ , called the Berwald L-lift of Γ . Then we particularize our study to the L-lift of a conservative L-connection. This L-lift enjoys some interesting properties. We finally deduce various identities concerning the curvature tensors of such a lift. This generalizes similar identities found in [9].

1. Notations and Preliminaries

The following notations will be used throughout the paper:
M: a differentiable manifold of class C^∞ and of finite dimension.
$T(M)$: the tangent bundle of M.
$\mathfrak{X}(M)$: the Lie algebra of vector fields on M.
J: the natural almost-tangent structure on $T(M)$ ([8] and [5]).
i_K: the interior product with respect to the vector form K.

All geometric objects considered in this paper are supposed to be of class C^∞. The formalism of Frölicher-Nijenhuis [2] will be our fundamental tool. The whole work is based on the approach developed in [10], which relies, in turn, on [1] and [3]. We give here a brief account of such approach.

Let M be a C^∞ manifold of dimension $2n$. Let L be a vector 1-form on M of constant rank n and such that $[L, L] = 0$ and that $Im(L_z) = Ker(L_z)$ for all $z \in M$. It follows that $L^2 = 0$ and $[C, L] = -L$, where C is the canonical vector field on M [10]. We call the linear space $Im(L_z) = Ker(L_z)$ the *vertical space* of M at z and denote it by $V_z(M)$; and as a vector bundle, we write $V(M)$.

A vector form K on M is said to be *homogeneous* of degree r if $[C, K] = (r-1)K$. It is called *L-semibasic* if $LK = 0$ and $i_X L = 0$ for all $X \in V(M)$. A vector field $S \in \mathfrak{X}(M)$ is said to be an *L-semispray* on M if $LS = C$. An L-semispray is an *L-spray* if it is homogeneous of degree 2. The *potential* of an L-semibasic vector k-form K on M is the L-semibasic vector $(k-1)$-form defined by $K^\circ = i_S K$, where S is an arbitrary L-semispray.

A vector 1-form Γ on M is called a nonlinear L-connection, or simply an *L-connection*, on M if $L\Gamma = L$ and $\Gamma L = -L$. An L-connection Γ on M is said to be *homogeneous* if it is homogeneous of degree 1 as a vector 1-form. A homogeneous L-connection Γ on M is said to be *conservative* if there exists an L-spray S on M such that $\Gamma = [L, S]$. An L-connection Γ on M defines an almost-product structure on M such that for all $z \in M$, the eigenspace of Γ_z corresponding to the eigenvalue (-1) coincides with the vertical space $V_z(M)$. The *vertical and horizontal projectors* of Γ are defined respectively by $v = \frac{1}{2}(I - \Gamma)$ and $h = \frac{1}{2}(I + \Gamma)$ and we thus have the decomposition $T_z(M) = V_z(M) \oplus H_z(M)$ for all $z \in M$, where $H_z(M) = Im(h_z)$: the *horizontal space* at z.

Let Γ be an L-connection on M. The *torsion* of Γ is the L-semibasic vector 2-form $T = \frac{1}{2}[L, \Gamma]$. The *strong torsion* of Γ is the L-semibasic vector 1-form $t = T^\circ + [C, v]$. The strong torsion of Γ vanishes if, and only if, Γ is homogeneous with no torsion. The *curvature* of Γ is the L-semibasic vector 2-form $\Omega = -\frac{1}{2}[h, h]$. An L-connection Γ on M is *strongly flat* if both its curvature and strong torsion vanish. The vector 1-form F on M defined by $FL = h$ and $Fh = -L$ defines an almost-complex structure on M such that $LF = v$. F is called the *almost-complex structure associated with* Γ .

2. Induced L-Connections

In this section we show that an L-regular linear connection D on M induces an L-connection on M and we study such L-connection in relation with D.

Definition 2.1. [7] *Let D be a linear connection on M. The map*

$$K : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M) : X \longmapsto D_X C$$

is called the connection map associated with D. That is, $K = DC$.

Definition 2.2. *A linear connection D on M is said to be L-almost-tangent if $DL = 0$; that is if*

$$D_X LY = LD_X Y \quad \forall X, Y \in \mathfrak{X}(M).$$

For an L-almost-tangent connection, $K(X)$ is vertical for every $X \in \mathfrak{X}(M)$.

Definition 2.3. *A linear connection D on M is said to be L-regular if it satisfies the conditions*:
(a) *D is L-slmost-tangent,*
(b) *the map $V(M) \longrightarrow V(M) : X \longmapsto K(X)$ is an isomorphism on $V(M)$.*
The inverse of this map will be denoted by φ.

For an L-regular linear connection, $\varphi \circ K = K \circ \varphi = I$ on $V(M)$.

Let D be an L-regular linear connection on M. By definition, the vertical component vX of $X \in \mathfrak{X}(M)$ is

$$vX = \varphi(K(X))$$

and the horizontal component hX is

$$hX = X - \varphi(K(X))$$

Hence, any vector field $X \in \mathfrak{X}(M)$ can be written as $X = vX + hX$ and we have the decomposition of $T(M)$:

$$T(M) = V(M) \oplus H(M),$$

where $H(M)$ is the vector bundle of horizontal vectors.

The vertical and horizontal projectors v and h are thus given by:

$$v = \varphi \circ K, \qquad h = I - \varphi \circ K \tag{2.1}$$

One can easily show that:

$$Lv = 0, \quad vL = L, \quad Lh = L, \quad hL = 0 \tag{2.2}$$

$$K(vX) = K(X), \quad K(hX) = 0 \quad \forall X \in \mathfrak{X}(M) \tag{2.3}$$

Lemma 2.4. *If $\mathbf{T}$ and $\mathbf{R}$ are the torsion and curvature tensors of an L-almost-tangent connection D on M, respectively, then*
(a) $\mathbf{T}(LX, LY) = L\mathbf{T}(LX, Y) + L\mathbf{T}(X, LY)$
(b) $\mathbf{R}(X, Y)LZ = L\mathbf{R}(X, Y)Z$
for every $X, Y, Z \in \mathfrak{X}(M)$.

Proof. (a) follows from the fact that D is L-almost-tangent and that $L^2 = 0 = [L, L]$.
(b) is a direct consequence of the L-almost-tangency of D. $\square$

Let D be an L-regular linear connection on M. Following Tamnou [8], we will define on M an almost-complex and an almost-product structures using L and the horizontal projector h associated with D.

Define the vector 1-form G on M by

$$G(LX) = -hX, \quad G(hX) = LX \qquad \forall X \in \mathfrak{X}(M) \tag{2.4}$$

Clearly, $G^2 = -I$ and so G is an almost-complex structure on M.

Using equations (2.2) and (2.4) together with the properties of L, v and h, one can prove

Proposition 2.5. *The almost-complex structure G has the following properties*:
(a) $GL = -h, \ Gh = L,$ $\quad$ (b) $LG = -v,$
(c) $Gv = hG = G - L,$ $\quad$ (d) $vG = G - Gv = L,$
(e) $GL + LG = -I,$ $\quad$ (f) $Gh + hG = G.$

Again, define the vector 1-form H on M by

$$H(LX) = hX, \quad H(hX) = LX \qquad \forall X \in \mathfrak{X}(M) \tag{2.5}$$

Cleary, $H^2 = I$ and so H is an almost-product structure on M.

Using equations (2.2) and (2.5) together with Proposition 2.5 and the properties of L, v, h and G, one can prove the analogue of Proposition 2.5 for H:

Proposition 2.6. *The almost-product structure H has the following properties*:
(a) $HL = h, \ Hh = L,$ $\quad$ (b) $LH = v,$
(c) $Hv = hH = H - L = -hG,$ $\quad$ (d) $vH = H - Hv = L,$
(e) $HL + LH = I,$ $\quad$ (f) $Hh + hH = H,$
(g) $GH = -HG,$ $\quad$ (h) $G + H = 2L.$

The above Properties (c), (g) and (h) above relate the two structures G and H.

The concept of almost-quaternionian structure in the next result is taken in the sense of Libermann [6].

Proposition 2.7. *The pair (G, H) defines an almost-quaternionian structure on (M, L, D).*

In fact, $G^2 = -H^2 = -I$ and $GH + HG = 0$.

Now, we define another almost-product structure Γ, of extreme importance, in terms of the two structures G and H.

Proposition 2.8. *The vector 1-form $\Gamma = HG$ is an almost-product structure on M.*

Proof. Using Proposition 2.6 and the fact that $G^2 = -I$, $H^2 = I$, we get
$\Gamma^2 = (HG)(HG) = H(GH)G = -H(HG)G = -H^2G^2 = I.$ $\square$

Theorem 2.9. *To each L-regular linear connection D on M there is associated a unique L-connection Γ on M given by $\Gamma = HG$, where G and H are defined respectively by* (2.4) *and* (2.5).

Proof. Using Propositions 2.5 and 2.6, we get:
$L\Gamma = L(HG) = (LH)G = vG = L$ and $\Gamma L = (HG)L = H(GL) = -Hh = -L$. Hence, Γ is an L-connection on M. Uniqueness is straightforward. $\square$

Definition 2.10. *Let D be an L-regular linear connection on M. The L-connection Γ defined in theorem 2.9 is said to be the L-connection on M induced by D.*

The next result expresses Γ in an explicit form in terms of the connection map K of Definition 2.1.

Theorem 2.11. *Let D be an L-regular linear connection on M. The L-connection Γ induced by D is expressed in the form*

$$\Gamma = I - 2\varphi \circ K, \tag{2.6}$$

where K is the connection map associated with D and φ is the inverse map of the restriction of K on $V(M)$.

Proof. Using Propositions 2.5 and 2.6, we get:

$$H + G = 2L \Longrightarrow \Gamma - I = 2LG \Longrightarrow \Gamma - v - h = -2v \Longrightarrow \Gamma = h - v.$$

Now, for every $X \in \mathfrak{X}(M)$, $\Gamma X = hX - vX = X - 2\varphi(K(X)) = (I - 2\varphi \circ K)X$; by virtue of (2.1). Hence (2.6) holds. $\square$

Corollary 2.12. *We have*
(a) $\Gamma = h - v$.
(b) $\Gamma h = h\Gamma = h, \quad \Gamma v = v\Gamma = -v$.

Corollary 2.13. *The vertical and horizontal projectors of Γ coincide with the vertical and horizontal projectors of D, respectively.*

In fact, we have , $\frac{1}{2}(I - \Gamma) = \frac{1}{2}(I - I + 2\varphi \circ K) = \varphi \circ K = v$, by (2.1) and (2.6). Similarly, $\frac{1}{2}(I + \Gamma) = h$.

Remark 2.14. When $M = T(N)$; N being a differentiable manifold of dimension n, and $L = J$, the induced nonlinear connection on M defined by Grifone [4] is retrieved as a special case of Theorem 2.11.

Throughout the remaining part of this section, D will denote an L-regular linear connection on M, K its connection map and Γ the L-connection on M induced by D.

Proposition 2.15. *The L-connection Γ is homogeneous if, and only if, K is homogeneous of degree one.*

Proof. We have by (2.1),

$$[C,v]=[C,\varphi\circ K]=\varphi\circ[C,K]+[C,\varphi]\circ K \tag{2.7}$$

We calculate the last term of (2.7). For every $X\in\mathfrak{X}(M)$,

$$[C,\varphi]K(X)=[C,(\varphi\circ K)X]-\varphi[C,K(X)]$$

Since $K(X)$ and $[C,(\varphi\circ K)X]$ are vertical and since $\varphi\circ K=K\circ\varphi=I$ on the vertical bundle, then

$$[C,\varphi]K(X)=(\varphi\circ K)[C,(\varphi\circ K)X]-\varphi[C,(K\circ\varphi)K(X)]=-\varphi([C,K](\varphi\circ K)X).$$

Hence, we obtain

$$[C,\varphi]\circ K=-\varphi\circ[C,K]\circ\varphi\circ K \tag{2.8}$$

It follows from (2.7) and (2.8) that $[C,v]=\varphi\circ[C,K]\circ(I-\varphi\circ K)$. Then, by (2.1),

$$[C,v]=\varphi\circ[C,K]\circ h,$$

from which $[C,\Gamma]=0\Longleftrightarrow[C,K]=0$. $\Box$

Definition 2.16. *The connection D is said to be reducible if $D\Gamma=0$.*

Clearly, $D\Gamma=0$ if, and only if, $Dh=Dv$

Lemma 2.17. *Let F be the almost-complex structure associated with Γ . A sufficient condition for D to be reducible is that $DF=0$.*

Proof. Corollary 2.12 and the definition of F are used in the proof.
For every $X,Y\in\mathfrak{X}(M)$, we have

$$\begin{aligned}
D_X\Gamma Y&=D_Xh\Gamma Y+D_Xv\Gamma Y=D_XhY-D_XvY=D_XFLY-D_XvY\\
&=FD_XLY-D_XvY,\\
\Gamma D_XY&=\Gamma D_XhY+\Gamma D_XvY=\Gamma D_XFLY+\Gamma D_XvY\\
&=\Gamma FD_XLY+\Gamma D_XvY=FD_XLY-D_XvY,
\end{aligned}$$

since FD_XLY is horizontal and D_XvY is vertical. Hence the result. $\Box$

The condition of Lemma 2.17 will be shown later to be necessary (Proposition 2.19 below).

Theorem 2.18. *Let $\overline{D}$ be an L-regular linear connection on the vector bundle $V(M)\longrightarrow M$. There exists a unique reducible connection D on M whose restriction to $V(M)$ coincides with $\overline{D}$.*

Proof. It should first be noticed that for every $X\in\mathfrak{X}(M)$ the operator $\overline{D}_X$ acts on vertical vector fields while the operator D_X (to be determined) acts on vector fields on M.

Let $\overline{K}=\overline{D}C$ and $\overline{\varphi}$ the inverse of the isomorphism of $V(M)$ defined by the restriction of $\overline{K}$ to $V(M)$. The vector 1-form $\overline{\Gamma}=I-2\overline{\varphi}\circ\overline{K}$ is clearly an L-connection on M. Let F denote the almost-complex structure associated with $\overline{\Gamma}$. Set

$$D_XY=F\overline{D}_XLY+\overline{D}_XLFY. \tag{2.9}$$

D is a linear connection on M with the required properties. The proof follows the same lines as in [4] with the necessary modifications.

It is a simple matter to show that $DC = \overline{D}C$. Consequently, the L-connection Γ induced by D coincides with $\overline{\Gamma}$. (This justifies the use of the same symbol F for both almost-complex structurs associated with Γ and $\overline{\Gamma}$). $\Box$

Proposition 2.19. *The following assertions are equivalent*
(a) *D is reducible.*
(b) $DF = 0$.
(c) $Dv = Dh = 0$.

Proof.
(a)$\Longrightarrow$(b): follows from formula (2.9).
(b)$\Longrightarrow$(c): $Dv = D(LF) = LDF = 0, \quad Dh = D(FL) = FDL = 0$.
(c)$\Longrightarrow$(a): $D\Gamma = D(h - v) = Dh - Dv = 0$. $\Box$

Remark 2.20. If an L-regular linear connection on M is reducible, it is completely determined by its action on the vertical bundle.

In fact, $D_X hY = D_X FLY = FD_X LY$.

3. L-Lifts and L-Connections

We have seen that each L-regular linear connection on M induces canonically an L-connection on M. We shall investigate here the converse problem.

Definition 3.1. *A linear connection D on M is said to be L-normal if it satisfies the conditions*
(a) D is L-almost-tangent,
(b) $D_{LX}C = LX$ for all $X \in \mathfrak{X}(M)$.

An L-normal linear connection is clearly L-regular. In fact, the map $LX \longmapsto K(LX)$ in Definition 2.3 is the identity map, and so $\varphi = I_{V(M)}$.

Lemma 3.2. *Let D be an L-almost-tangent linear connection on M such that $D_C LX = L[C, X]$ for all $X \in \mathfrak{X}(M)$. The connection D is L-normal if, and only if, $\mathbf{T}(C, LX) = 0$, where $\mathbf{T}$ is the torsion of D.*

Proof. We have:

$$\begin{aligned}\mathbf{T}(C, LX) &= D_C LX - D_{LX}C - [C, LX] = L[C, X] - [C, LX] - D_{LX}C \\ &= [L, C]X - D_{LX}C = LX - D_{LX}C.\end{aligned}$$

Hence, $D_{LX}C = LX \Longleftrightarrow \mathbf{T}(C, LX) = 0$. $\Box$

Definition 3.3.
– *Let D be a given L-normal linear connection on M. The L-connection Γ on M induced by D is called the L-projection of D.*
– *Let Γ be a given L-connection on M. An L-normal linear connection D on M whose L-projection coincides with Γ is called an L-lift of Γ . If D is reducible, it is called a reducible L-lift of Γ .*

The following result shows (roughly) that there is associated a reducible L-lift to each L-connection on M.

Theorem 3.4. *Let Γ be an L-connection on M and let B be an L-semibasic vector 2-form on M such that $B^\circ + [C, h] = 0$. There exists a unique reducible L-lift D of Γ whose torsion satisfies $\mathbf{T}(LX, Y) = B(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$.*

Proof. Set

$$D_X Y = h[LY, F]X + L[vY, F]X + FB(X, Y) + B(X, FY) \tag{3.1}$$

where F is the almost-complex structure associated with Γ and v and h are respectively the vertical and horizontal projectors of Γ . The connection D defined by (3.1) is the required L-lift of Γ . The proof is similar to that of Theorem III,32 of [4]. $\square$

As $DF = 0$, the connection (3.1) is completely determined by (cf. Corollary 2.13):

$$\left.\begin{aligned} D_{LX} LY &= L[LX, Y] \\ D_{hX} LY &= v[hX, LY] + B(X, Y) \end{aligned}\right\} \tag{3.2}$$

or, again, by

$$D_X LY = L[vX, Y] + v[hX, LY] + B(X, Y) \tag{3.3}$$

Remark 3.5. If Γ is homogeneous, $[C, h] = 0$. Hence, there exists, for every homogeneous L-connection, a canonical reducible L-lift characterized by $\mathbf{T}(LX, Y) = 0$ for all $X, Y \in \mathfrak{X}(M)$.

This L-lift is called the Berwald L-lift of Γ .

Remark 3.6. If $M = T(N)$; N being of dimension n, and $L = J$, the reducible J-lift of a J-connection Γ on $T(N)$ is nothing but the lift of Γ introduced by Grifone [4]. If moreover Γ is homogeneous and we choose $B = 0$, the reducible J-lift of Γ coincides with the linear extension, in the sense of Theorem 2.18, of the usual Berwald connection. This justifies the adopted terminology.

In the remaining part of the present section, let Γ denote an L-connection on M and D its reducible L-lift corresponding to the L-semibasic vector 2-form B. Also, let T, t, Ω and F denote the torsion, strong torsion, curvature and associated almost-complx structure of Γ , respectively. Let $\mathbf{T}$ and $\mathbf{R}$ be the torsion and curvature tensors of the linear connection D, respectively.

Proposition 3.7. *The torsion $\mathbf{T}$ of the L-lift D of Γ is given, for all $X, Y \in \mathfrak{X}(M)$, by*

$$\mathbf{T}(X, Y) = (F \circ T + \Omega)(X, Y) + (i_F B)(X, Y) + 2FB(X, Y)$$

Proof. For all $X, Y \in \mathfrak{X}(M)$, we have

$$\mathbf{T}(X, Y) = \mathbf{T}(hX, hY) + \mathbf{T}(hX, LFY) + \mathbf{T}(LFX, hY), \tag{3.4}$$

since $\mathbf{T}(vX, vY) = B(FX, vY) = 0$; B being L-semibasic.

Using (3.2) and the properties of the tensors associated with Γ , we get after some calculations:

$$\mathbf{T}(hX, hY) = h^*[F, F](X, Y) + 2FB(X, Y), \tag{3.5}$$

$$\mathbf{T}(hX, LFY) = B(X, FY), \tag{3.6}$$

$$\mathbf{T}(LFX, hY) = B(FX, Y), \tag{3.7}$$

where $h^*[F,F](X,Y) = \frac{1}{2}[F,F](hX,hY)$.

Substituting (3.5), (3.6) and (3.7) into (3.4) and taking the fact that $h^*[F,F] = F \circ T + \Omega$ [10] into account, the result follows. $\square$

Theorem 3.8. *A necessary and sufficient condition for the existence of a symmetric L-lift of an L-connection Γ is that Γ be strongly flat.*

Proof. Suppose that there exists an L-lift of Γ such that $\mathbf{T} = 0$. Thus we have $0 = \mathbf{T}(LX, Y) = B(X, Y)$. Hence, by Proposition 3.7, $F \circ T + \Omega = 0$. But since $F \circ T$ has horizontal values while Ω has vertical values, then $T = 0$ and $\Omega = 0$. Now, as Γ is homogeneous ($[C, \Gamma] = 2B^\circ = 0$) and $T = 0$, it follows from Corollary 2 of [10] that $t = 0$. Hence, Γ is strongly flat.

Conversely, if Γ is strongly flat, then Γ is homogeneous [10] and the Berwald L-lift of Γ is evidently symmetric (cf. Remark 3.5 and (3.5)). $\square$

As the L-lift D of an L-connection Γ is reducible, the curvature tensor $\mathbf{R}$ of D is completely determined by the three semibasic tensors:

$$\begin{aligned} R(X,Y)Z &= \mathbf{R}(hX, hY)LZ \\ P(X,Y)Z &= \mathbf{R}(hX, LY)LZ \\ Q(X,Y)Z &= \mathbf{R}(LX, LY)LZ \end{aligned}$$

Using (3.2) and (3.3), the properties of the tensors associated with Γ and the fact that B is L-semibasic, we get after long calculations

Proposition 3.9. *The three curvature tensors R, P and Q of D are respectively given, for all $X, Y, Z \in \mathfrak{X}(M)$, by*

(a) $R(X,Y)Z = (D_{LZ}\Omega)(X,Y) + (D_{hY}B)(Z,X) - (D_{hX}B)(Z,Y)$
$\qquad + B(FB(Z,X),Y) - B(FB(Z,Y),X) + B(FT(X,Y),Z).$

(b) $P(X,Y)Z = (D_{LY}B)(Z,X) + v[hX, L[LY,Z]] + v[LZ,[hX,LY]]$
$\qquad - L[LY, F[hX,LZ]] - L[LZ, F[hX,LY]].$

(c) $Q(X,Y)Z = 0.$

4. Berwald L-Lifts of Homogeneous L-Connections

In this section, Γ will denote a **homogeneous** L-connection on M. The reducible L-lift of Γ characterized by $\mathbf{T}(LX,Y) = 0$, for all $X, Y \in \mathfrak{X}(M)$, is called the Berwald L-lift of Γ (cf. Remark 3.5).

By virtue of (3.2), the Berwald L-lift D is completely determined by:

$$\left.\begin{aligned} D_{LX}LY &= L[LX, Y] \\ D_{hX}LY &= v[hX, LY] \end{aligned}\right\} \tag{4.1}$$

or, again, by

$$D_X LY = L[vX, Y] + v[hX, LY] \tag{4.2}$$

Lemma 4.1. *The Berwald L-lift D of Γ is such that*
(a) $D_C LX = L[C, X]$.
(b) $[C, DLX] = D[C, LX]$.

Proof. (a) follows from the first formula of (4.1) by letting $X = S$; an arbitrary L-semispray.
(b) follows from (4.2), the properties of L and those of the tensors associated with Γ and from the Jacobi identity. $\square$

Remark 4.2. In view of the above lemma, as the Berwald L-lift D of Γ is reducible, D is an "extended connection of directions" in the sense of Grifone [4] (where $M = T(N)$ and $L = J$).

Proposition 4.3. *The torsion tensor of the Berwald L-lift of Γ is given by*

$$\mathbf{T} = F \circ T + \Omega$$

This result follows directly from Proposition 3.7.

Corollary 4.4. *If Γ is a conservative L-connection on M, then*

$$\mathbf{T} = \Omega$$

In fact, $T = 0$ for conservative L-connections [10].

Proposition 4.5. *The first curvature tensor of the Berwald L-lift D of Γ is given by*

$$R(X, Y)Z = (D_{LZ}\Omega)(X, Y) \tag{4.3}$$

This result follows immediately from Proposition 3.9.

Theorem 4.6. *For the Berwald L-lift D of Γ, we have*

$$R(X, Y)S = \Omega(X, Y), \tag{4.4}$$

where S is an arbitrary L-semispray on M.
Consequently, $R = 0$ if, and only if, $\Omega = 0$.

Proof. Setting $Z = S$ in (4.3), taking the fact that Ω is L-semibasic into account, we get

$$R(X, Y)S = (D_C\Omega)(X, Y) = D_C\Omega(X, Y) - \Omega(D_C hX, Y) - \Omega(X, D_C hY)$$

Using Lemma 4.1(a) together with (4.2), we get

$$\begin{aligned} R(X, Y)S &= L[C, F\Omega(X, Y)] - \Omega([C, X], Y) - \Omega(X, [C, Y]) \\ &= -[C, L]F\Omega(X, Y) + [C, \Omega(X, Y)] - \Omega([C, X], Y) - \Omega(X, [C, Y]) \\ &= LF\Omega(X, Y) + [C, \Omega](X, Y) \\ &= \Omega(X, Y);\ \Omega \text{ being homogeneous of degree 1 (since } \Gamma \text{ is).} \end{aligned}$$

Now, if $\Omega = 0$, then $R = 0$, by (4.3). Conversely, if $R = 0$, then $\Omega = 0$, by (4.4). (Note that we have shown, in the course of the proof, that $D_C\Omega = \Omega$.) $\Box$

For the rest of the paper, we consider the Berwald L-lift D of a **conservative** L-connection Γ on M.

As a conservative L-connection Γ on M is homogeneous with no torsion and is of the form $\Gamma = [L, S]$, we may combine Theorem 4.6 and Theorems 6, 7 and 9 of [10] to obtain the following result:

Theorem 4.7. *For the Berwald L-lift of a conservative L-connection on M, the following assertions are equivalent*
(a) $\Omega^\circ = 0$.
(b) $\Omega = 0$.
(c) $R = 0$.
(d) $[F, F] = 0$.
(e) *the horizontal distribution* $z \longmapsto H_z(M)$ *is completely integrable.*

As for all linear connections, the (classical) Bianchi's identities for D are given by:

$$\mathfrak{S}\, \mathbf{R}(X,Y)Z = \mathfrak{S}\, \{\mathbf{T}(\mathbf{T}(X,Y),Z) + (D_X\mathbf{T})(Y,Z)\},$$
$$\mathfrak{S}\, \{\mathbf{R}(\mathbf{T}(X,Y),Z) + (D_X\mathbf{R})(Y,Z)\} = 0,$$

where $\mathfrak{S}$ denotes the cyclic permutation of the vector fields X, Y and Z.

But since Γ is conservative, we have, by Corollary 4.4, $\mathbf{T} = \Omega$, which is L-semibasic. Thus the above identities reduce to:

$$\mathfrak{S}\, \mathbf{R}(X,Y)Z = \mathfrak{S}\, (D_X\Omega)(Y,Z) \tag{4.5}$$

$$\mathfrak{S}\, \{\mathbf{R}(\Omega(X,Y),Z) + (D_X\mathbf{R})(Y,Z)\} = 0 \tag{4.6}$$

These two identities give rise to the following useful identities.

Proposition 4.8. *For the Berwald L-lift of a conservative L-connection on M, we have for all* $X, Y, Z \in \mathfrak{X}(M)$:
(a) $\mathfrak{S}\, R(X,Y)Z = 0$.
(b) $\mathfrak{S}\, (D_{hX}R)(Y,Z) = \mathfrak{S}\, P(X, F\Omega(Y,Z))$.
(c) $(D_{LZ}R)(X,Y) = (D_{hY}P)(X,Z) - (D_{hX}P)(Y,Z)$.
(d) $(D_{LZ}P)(X,Y) = (D_{LY}P)(X,Z)$.
(e) $P(X,Y)Z = P(Y,X)Z = P(Z,X)Y$. *(P is symmetric in its three variables.)*

Sketch of the Proof.
(a) Compute (4.5) for hX, hY, hZ.
(b) Compute (4.6) for hX, hY, hZ.
(c) Compute (4.6) for hX, hY, LZ.
(d) Compute (4.6) for hX, LY, LZ.
(e) Compute (4.5) for hX, hY, LZ.
The calculations are too long but not difficult. So, we omit them. $\Box$

Corollary 4.9. *We have*
(a) $\mathfrak{S}\ (D_{hX}\Omega)(Y,Z) = 0.$
(b) $\mathfrak{S}\ (D_{LX}\Omega)(Y,Z) = 0.$
(c) $\mathfrak{S}\ (D_{LX}R)(Y,Z) = 0.$

Sketch of the Proof.
(a) follows from Proposition 4.8(a).
(b) follows from Proposition 4.8(a) and from (4.3).
(c) follows from Proposition 4.8(c) and (e). $\square$

Remark 4.10. The identities in Proposition 4.8 and Corollary 4.9 are similar to those found in [9]. Nevertheless, the context here is more general and the scope of validity is much wider. In fact, the above identities are valid for the large class of L-lifts of conservative L-connections, while the identities in [9] are valid only for the Berwald connection as a J-lift of the canonical connection associated with a Finsler space.

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